Computer graphics III – Multiple Importance Sampling

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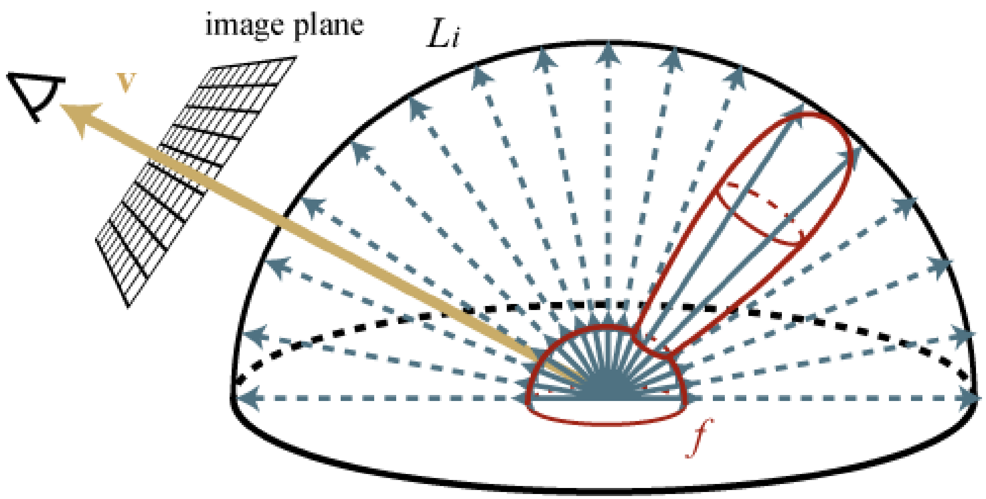
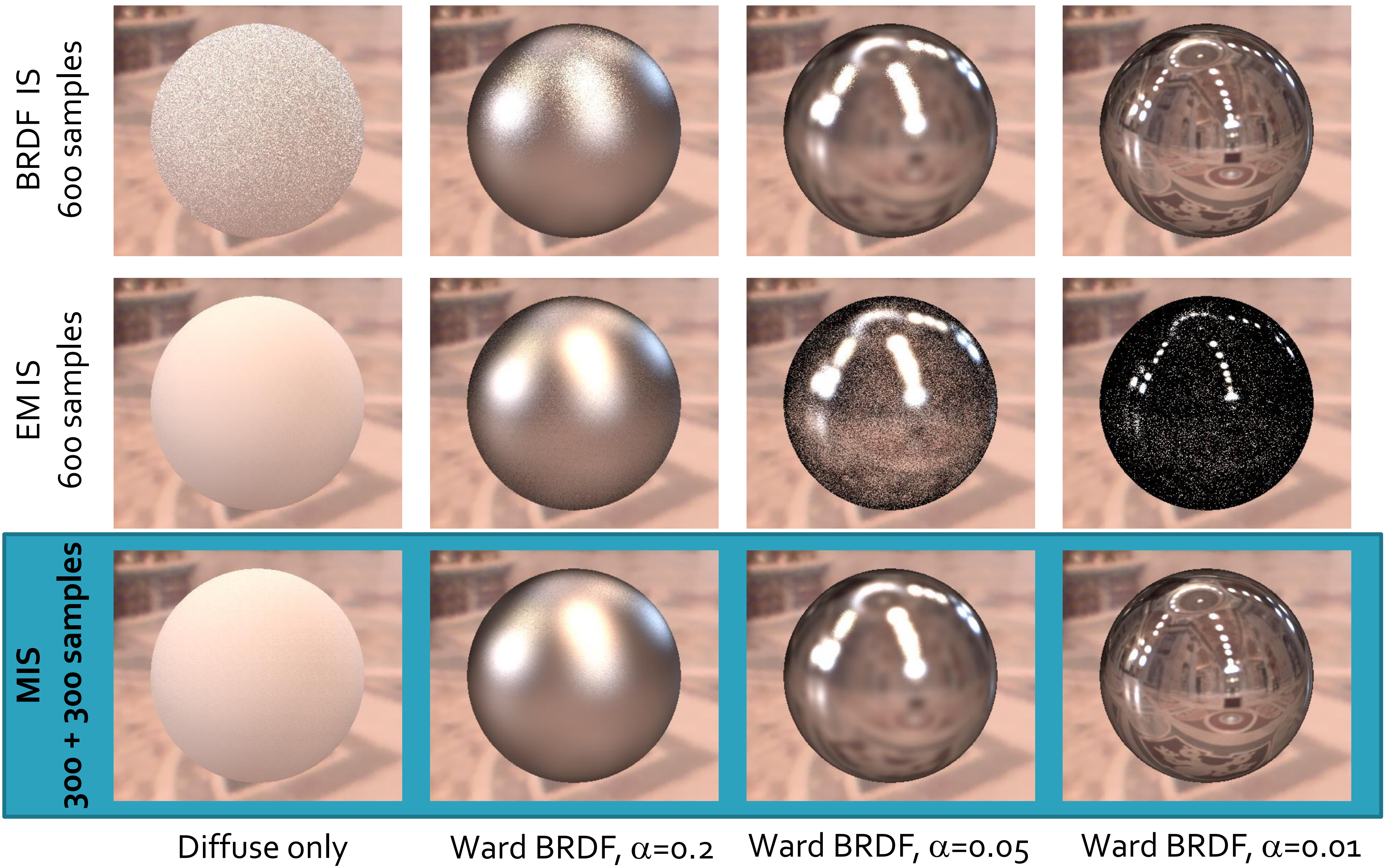
Sampling of environment lighting



$$L_r(\omega_o) = \int_{H(\mathbf{x})} L_i(\omega_i) \cdot f_r(\omega_i \rightarrow \omega_o) \cdot \cos\theta_i \, d\omega_i$$

Sampling of environment lighting

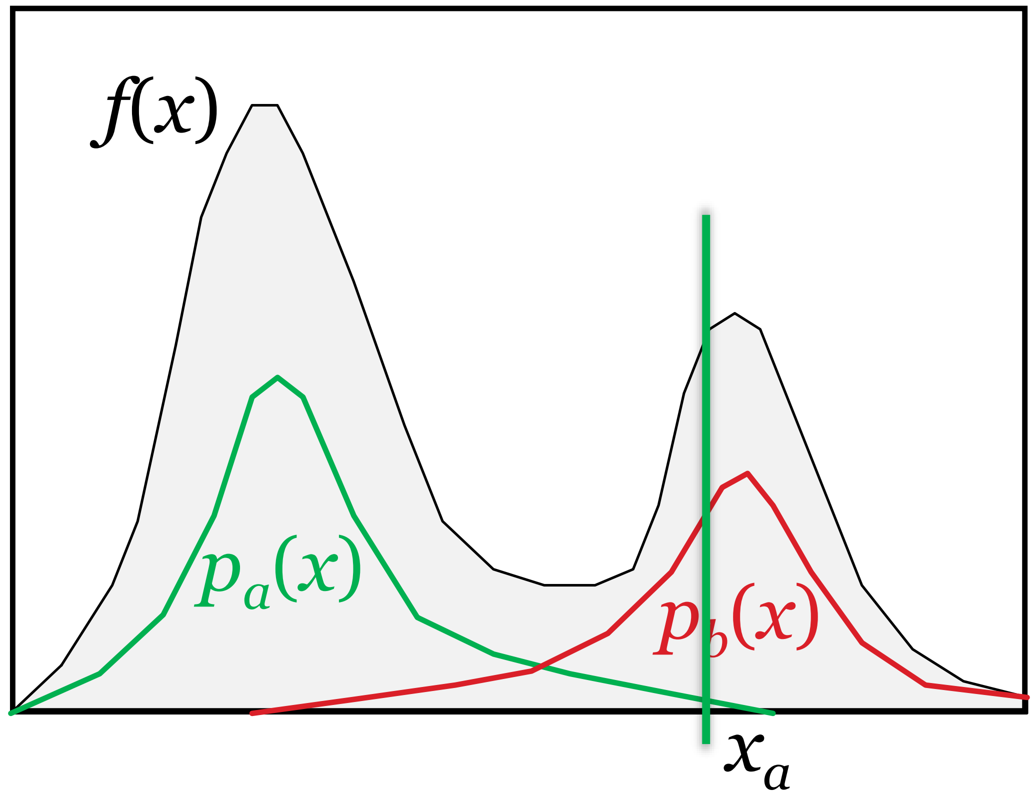


Sampling of environment lighting

- Two different sampling strategies for generating the incident direction ω_i

1. **BRDF-proportional sampling -** $p_a(\omega_i)$
2. **Environment map-proportional sampling -** $p_b(\omega_i)$

What is wrong with using either of the two strategies alone?



Notes on the previous slide

- We have a complex multimodal integrand $f(x)$ that we want to numerically integrate using a MC method with importance sampling.
- Unfortunately, we do not have a PDF that would mimic the integrand in the entire domain.
- Instead, we can draw the sample from two different PDFs, p_a and p_b each of which is a good match for the integrand under different conditions – i.e. in different part of the domain.
- However, the estimators corresponding to these two PDFs have extremely high variance – shown on the slide.
- We can use Multiple Importance Sampling (MIS) to combine the sampling techniques corresponding to the two PDFs into a single, robust, combined technique.
- The MIS procedure is extremely simple: sample from both techniques p_a and p_b , and then takes the sample from the selected distribution.

- This estimator is really powerful at suppressing outlier samples such as those that you would obtain by picking x_from the tail of p_a, where $f(x)$ might still be large.
- Without having p_b at our disposal, we would be dividing the large $f(x)$ by the small $p_a(x)$, producing an outlier.
- However, the combined technique has a much higher chance of producing this particular x (because it can sample it also from p_b), so the combined estimator divides $f(x)$ by $[p_a(x) + p_b(x)] / 2$, which yields a much more reasonable sample value.

- I want to note that what I'm showing here is called the "balance heuristic" and is a part of a wider theory on weighted combinations of estimators proposed by Veach and Guibas.

Multiple Importance Sampling

First for general estimators, so please forget the direct illumination problem for a short while.

- Start with enviro example

Multiple Importance Sampling

- Given n sampling techniques (i.e. pdfs) $p_1(x), .. , p_n(x)$
- We take n_i samples $X_{i,1}, .. , X_{i,n_i}$ from each technique
- **Combined estimator**

Combination weights (different for each sample)

$$F = \sum_{i=1}^{n} \frac{1}{n_i} \sum_{j=1}^{n_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

sampling techniques

samples from individual techniques

Unbiasedness of the combined estimator

$$E[F] = \ldots = \int \left[\sum_{i=1}^{n} w_i(x) \right] f(x)\,\mathrm{d}x \equiv \int f(x)$$

- Condition on the weighting functions

$$\forall x: \quad \sum_{i=1}^{n} w_i(x) = 1$$

Choice of the weighting functions

- **Objective:** minimize the variance of the combined estimator

1. Arithmetic average (very bad combination)

$$w_i(x) = \frac{1}{n}$$

2. **Balance heuristic** (very good combination)
   - ....

Balance heuristic

- Combination weights

$$\hat{w}_i(\mathbf{x}) = \frac{n_i p_i(\mathbf{x})}{\sum_k n_k p_k(\mathbf{x})}$$

- Resulting estimator (after plugging the weights)

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{f(X_{i,j})}{\sum_k n_k p_k(X_{i,j})},$$

  - The contribution of a sample does not depend on which technique (pdf) it came from
  - Effectively, the sample is drawn from a weighted average of the individual pdfs – as can be seen from the form of the estimator

Balance heuristic

- The balance heuristic **is almost optimal**
  - No other weighting has variance much lower than the balance heuristic

Direct illumination calculation using MIS

Application of MIS to environment light sampling

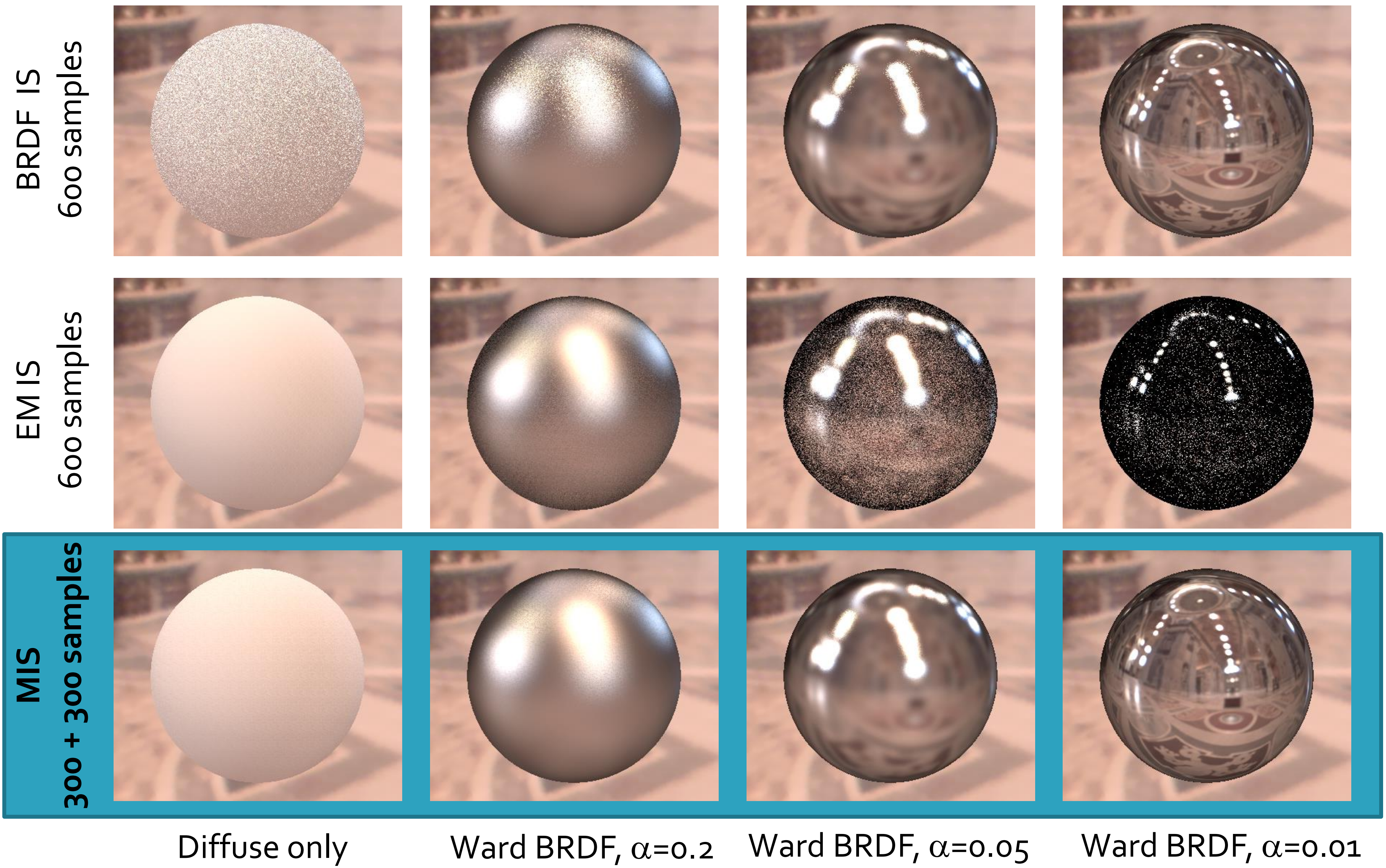
- Two sampling strategies for generating the incident direction ω_i

1. **BRDF-proportional sampling -** $p_a(\omega_i)$

2. **Environment map-proportional sampling -** $p_b(\omega_i)$

- Mindlessly plug $p_a(\omega_i)$ and $p_b(\omega_i)$ into the general formulas above

MIS applied to enviro sampling



Area light sampling – Motivation

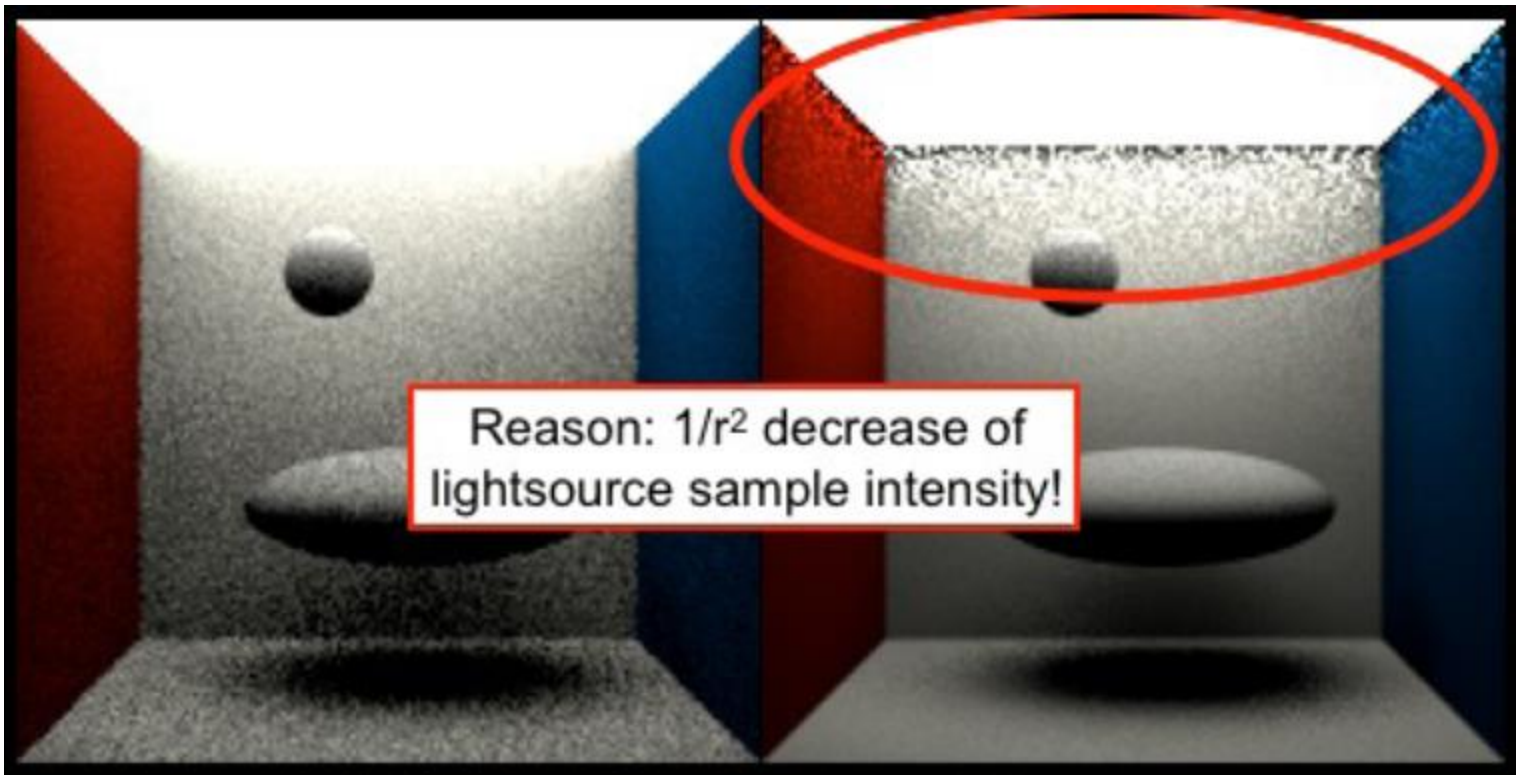
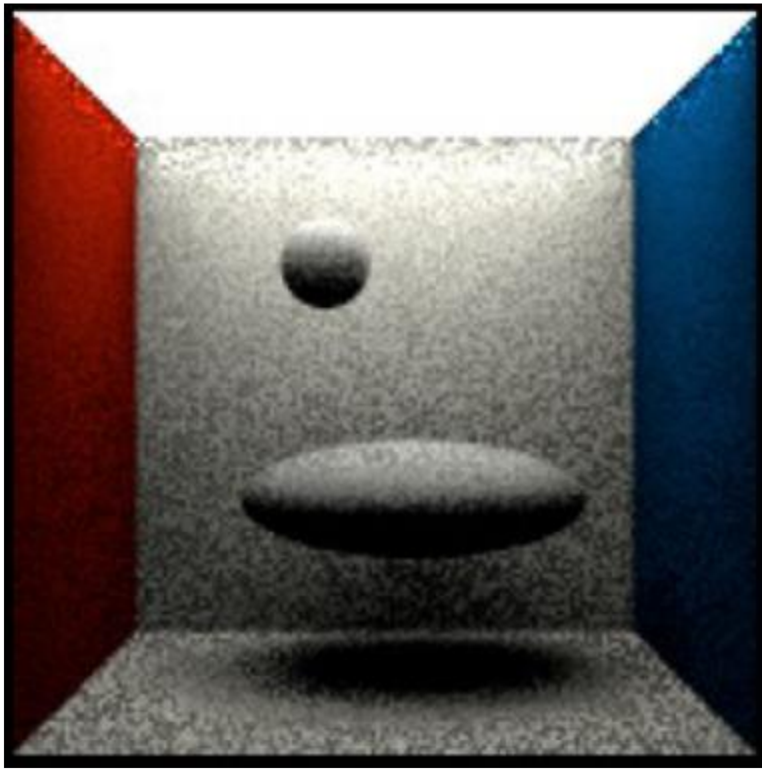


Image: Alexander Wilkie

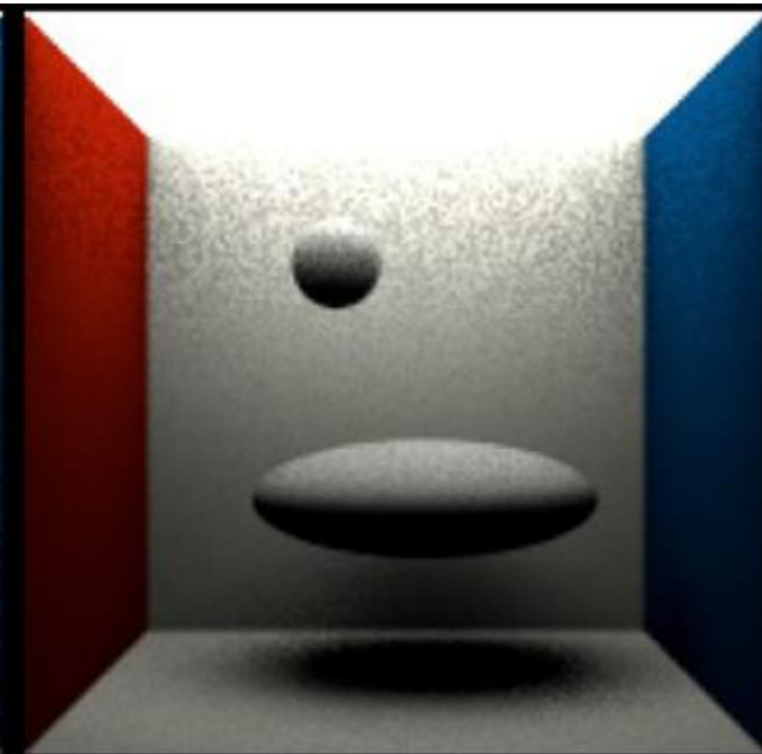
Sampling technique (pdf) p_a:
BRDF sampling

Sampling technique (pdf) p_b:
Light source area sampling

MIS-based combination



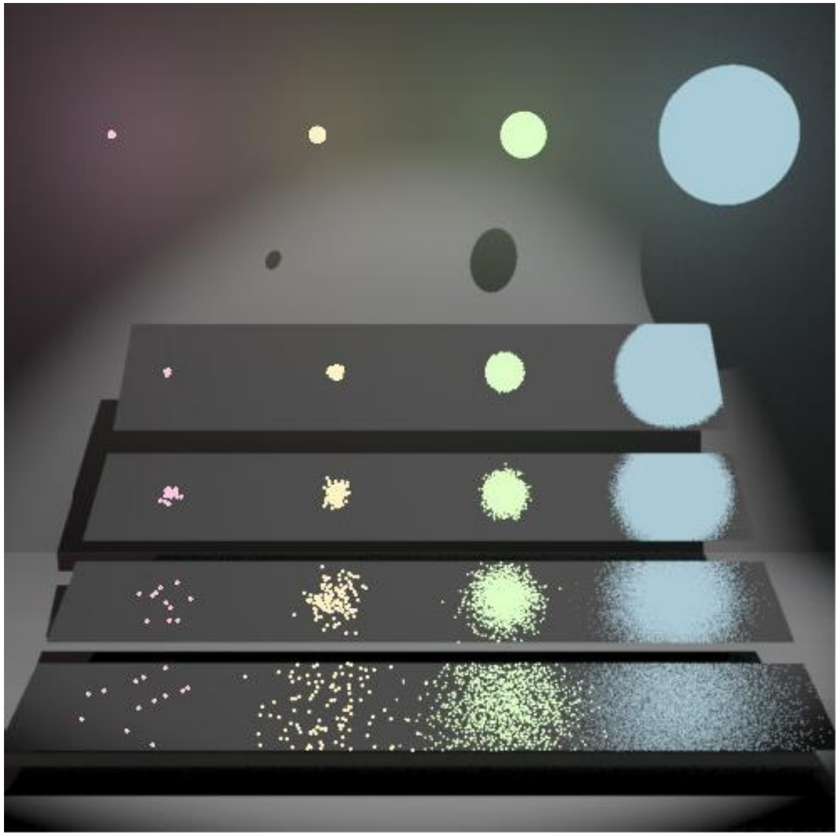
Arithmetic average
Preserves **bad** properties
of both techniques



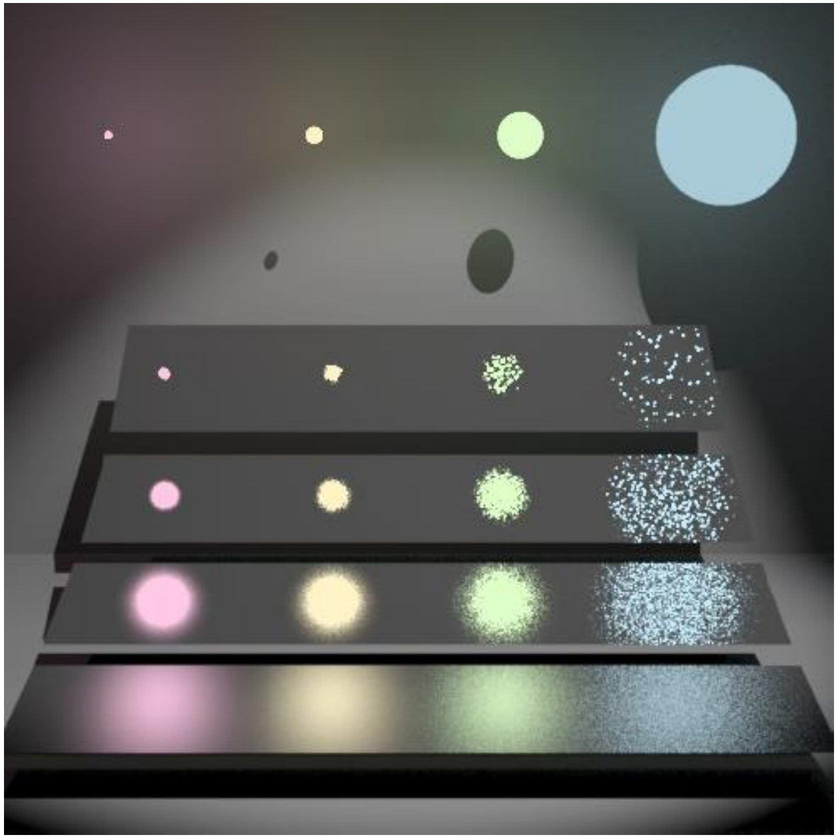
MIS w/ the balance heuristic
Bingo!!!

Image: Alexander Wilkie

Area light sampling – Classic Veach's example



BRDF proportional sampling



Light source area sampling

Images: Eric Veach

MIS-based combination

- **Multiple importance sampling** & **Balance heuristic** (Veach & Guibas, 95)

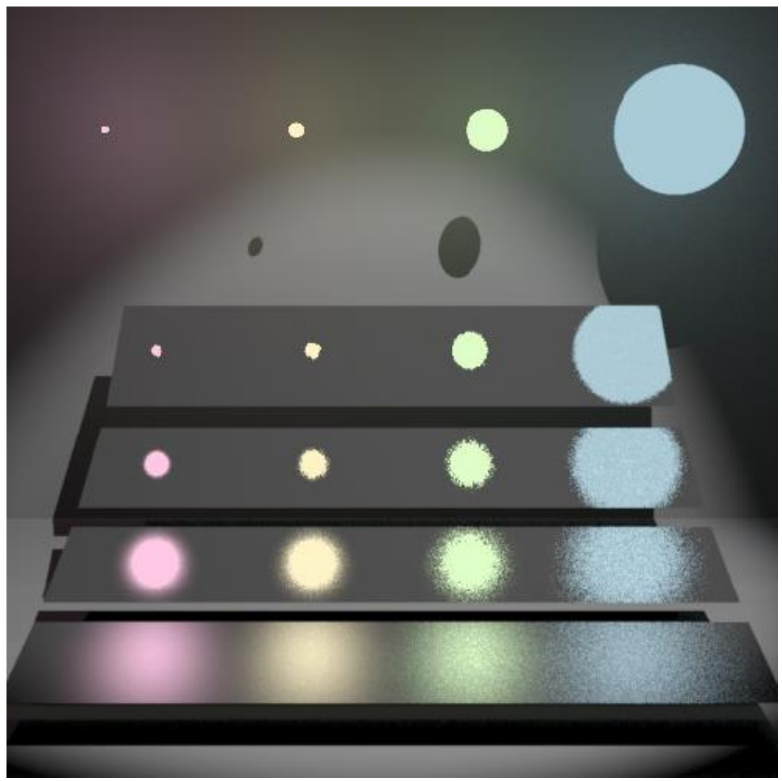


Image: Eric Veach

Area light sampling – sampling strategies

- Two sampling strategies – as for enviro maps
  1. **BRDF-proportional sampling**
  2. **Light source area sampling**

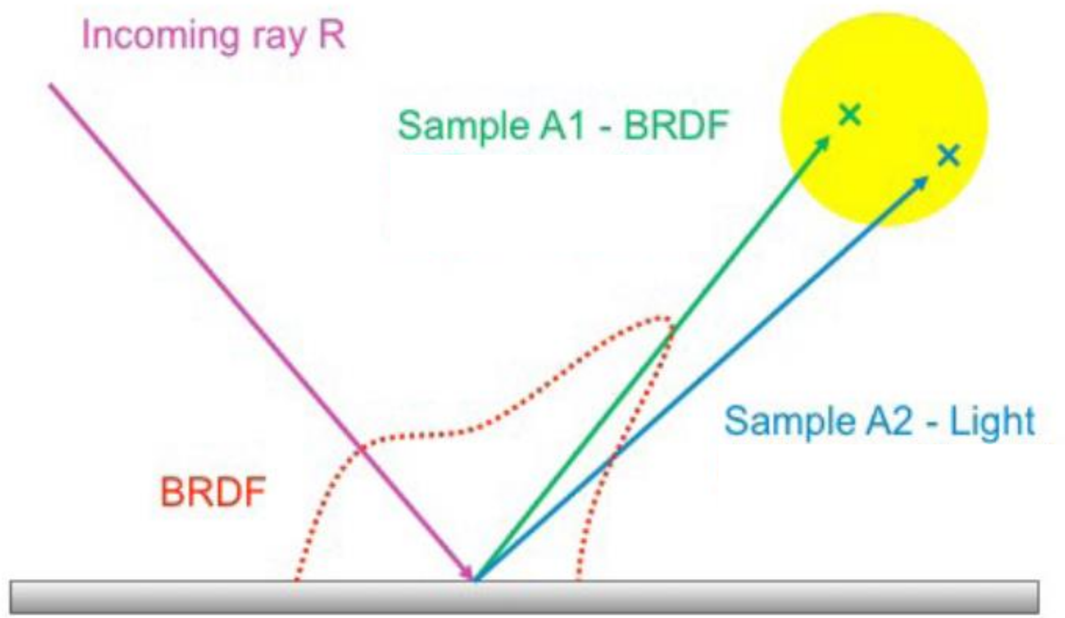


Image: Alexander Wilkie

Direct illumination: Two strategies

- **BRDF proportional sampling**
  - Better for large light sources and/or highly glossy BRDFs
  - The probability of hitting a small light source is small -> high variance, noise
- **Light source area sampling**
  - Better for smaller light sources
  - It is the only possible strategy for point sources
  - For large sources, many samples are generated outside the BRDF lobe -> high variance, noise

Direct illumination: Two strategies

- Which strategy should we choose?
  - **Both!**

- Both strategies estimate the same quantity $L_r(\mathbf{x}, \omega_o)$
  - A mere sum would estimate 2 x $L_r(\mathbf{x}, \omega_o)$, which is wrong

- We need a weighted average of the techniques, but **how to choose the weights**? => MIS

MIS weight calculation

Sample weight for BRDF sampling

$$w_a(\omega_j) = \frac{p_a(\omega_j)}{p_a(\omega_j) + p_b(\omega_j)}$$

PDF for BRDF sampling

PDF with which the direction ω_j would have been generated, if we used light source area sampling

Example PDFs

- **BRDF sampling: $p_a(\omega)$**
  - Depends on the BRDF, e.g. for a Lambertian BRDF:

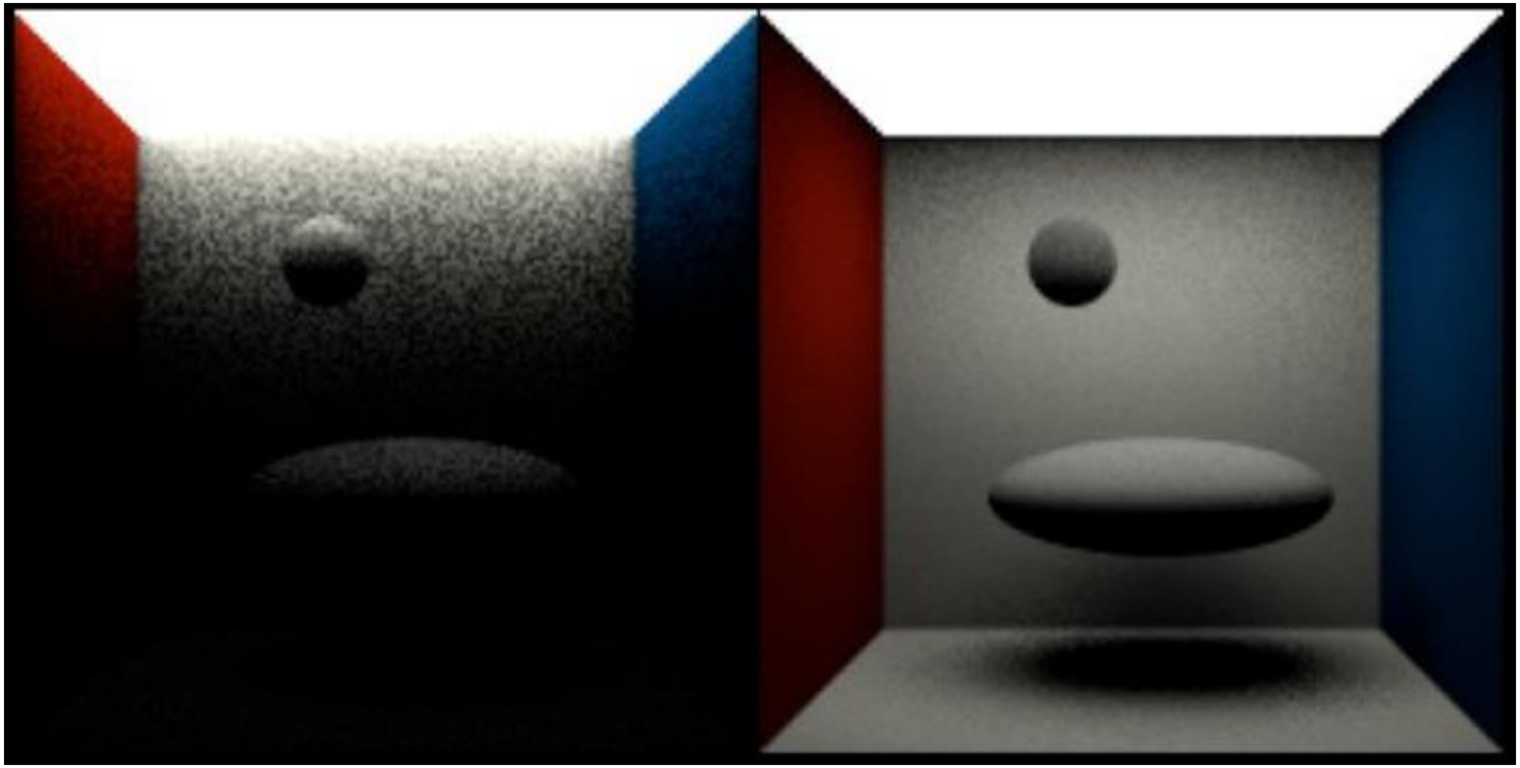
$$p_a(\omega) = \frac{\cos\theta_{\mathbf{x}}}{\pi}$$

- **Light source area sampling: $p_b(\omega)$**

$$p_b(\omega) = \frac{1}{|A|} \frac{||\mathbf{x} - \mathbf{y}||^2}{\cos\theta_{\mathbf{y}}}$$

Conversion of the uniform pdf 1/|A| from the area measure (dA) to the solid angle measure (dω)

Contributions of the sampling techniques

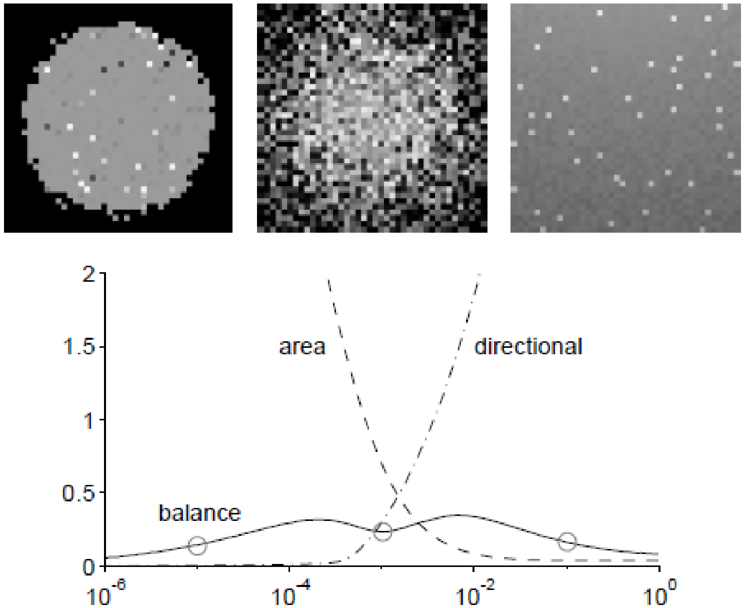


w_a * BRDF sampling **w_b * light source area sampling**

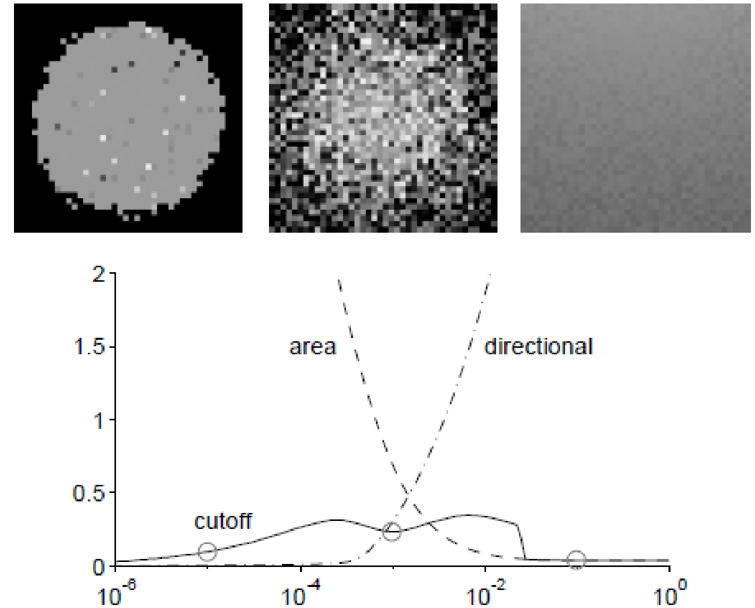
Image: Alexander Wilkie

Alternative combination heuristics

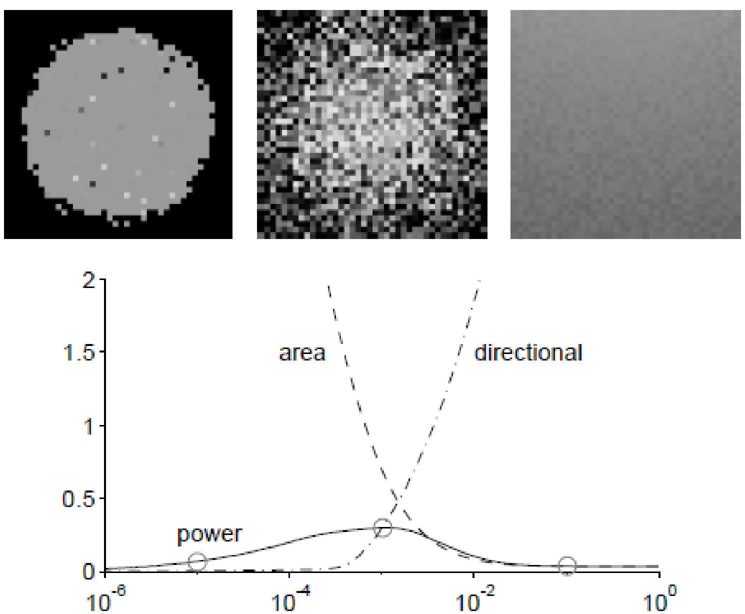
- “**Low variance problems**”
- Whenever one sampling technique yields a very low variance estimator, balance heuristic can be suboptimal
- “Power heuristic” or other heuristics can be better in such a case – see next slide



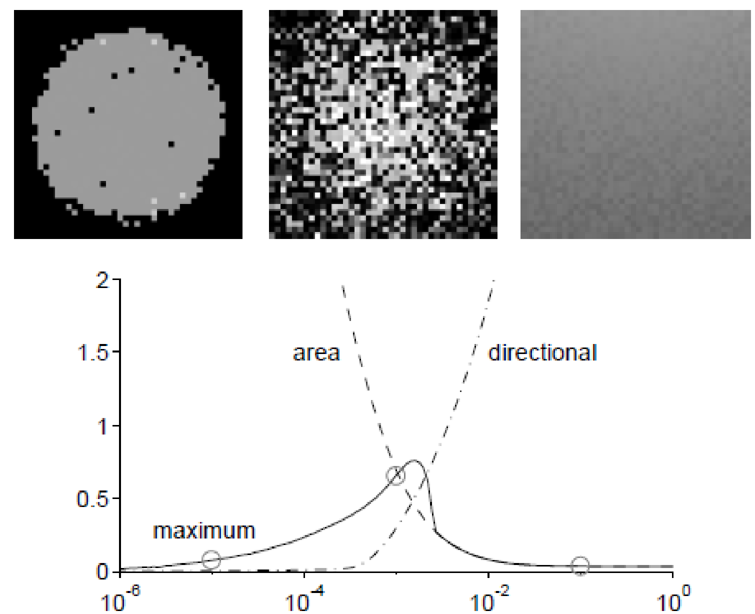
(a) The balance heuristic.



(b) The cutoff heuristic ($\alpha = 0.1$).



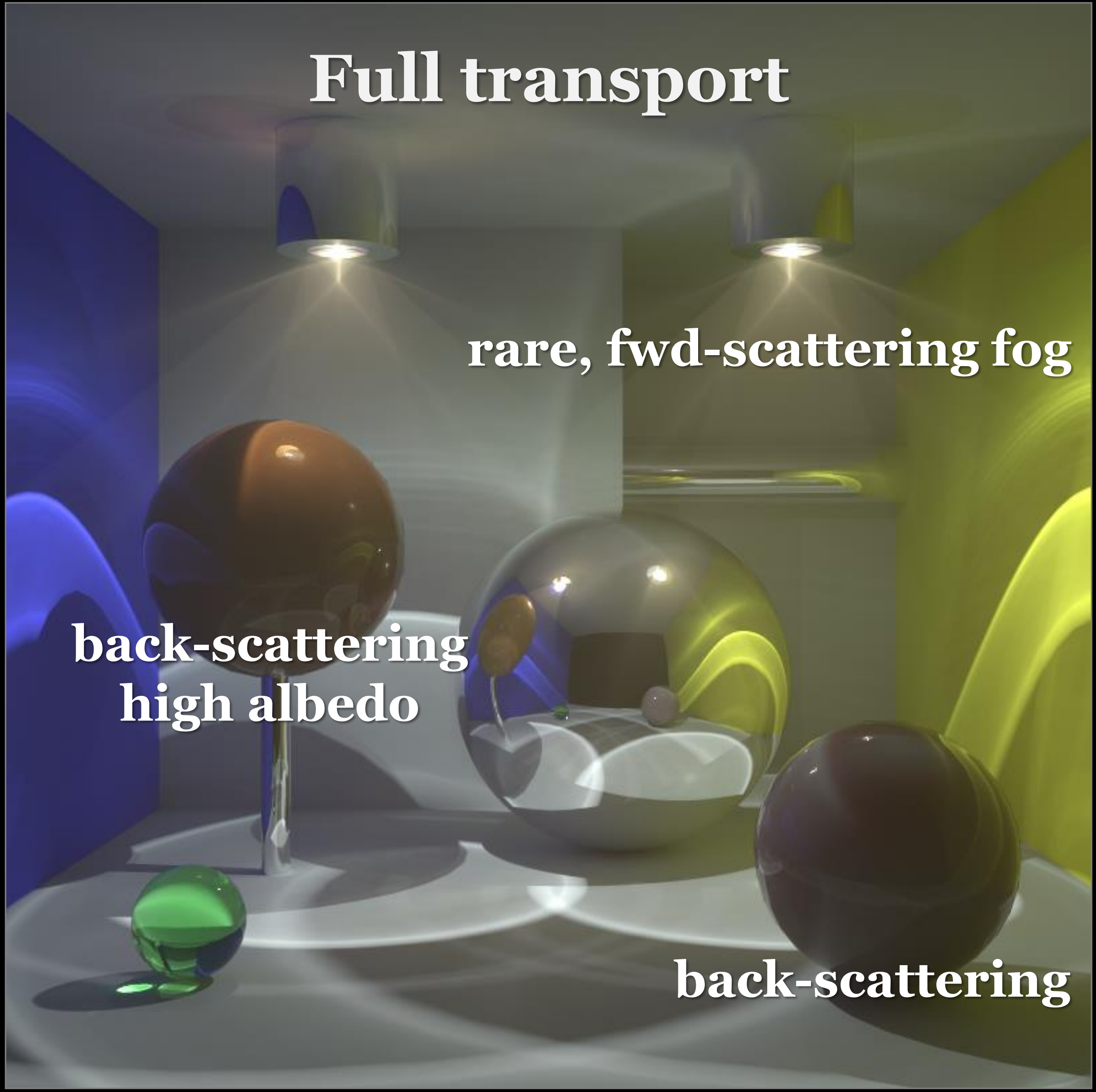
(c) The power heuristic ($\beta = 2$).



(d) The maximum heuristic.

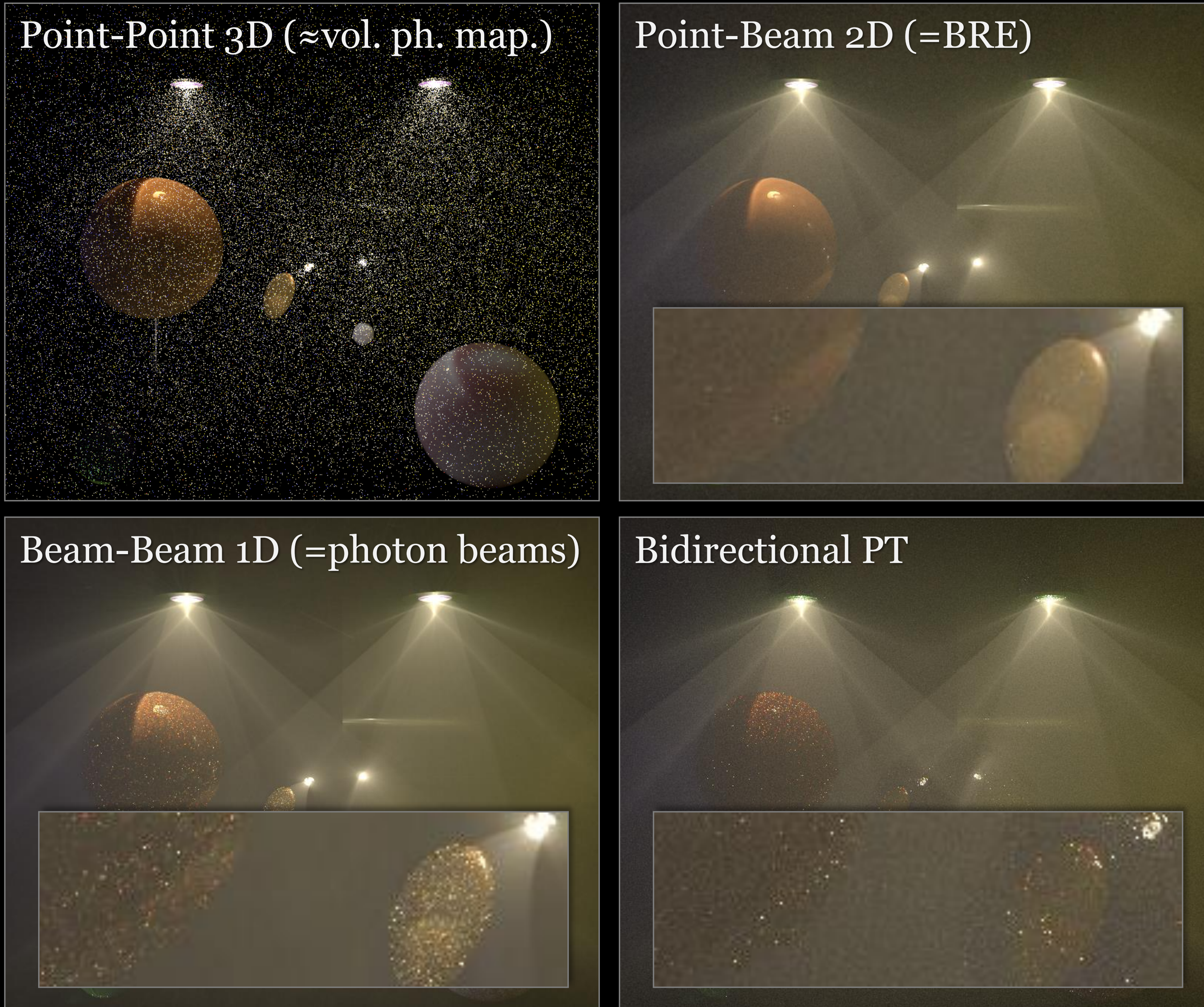
Other examples of MIS applications

In the following we apply MIS to combine full path sampling techniques for calculating light transport in participating media.

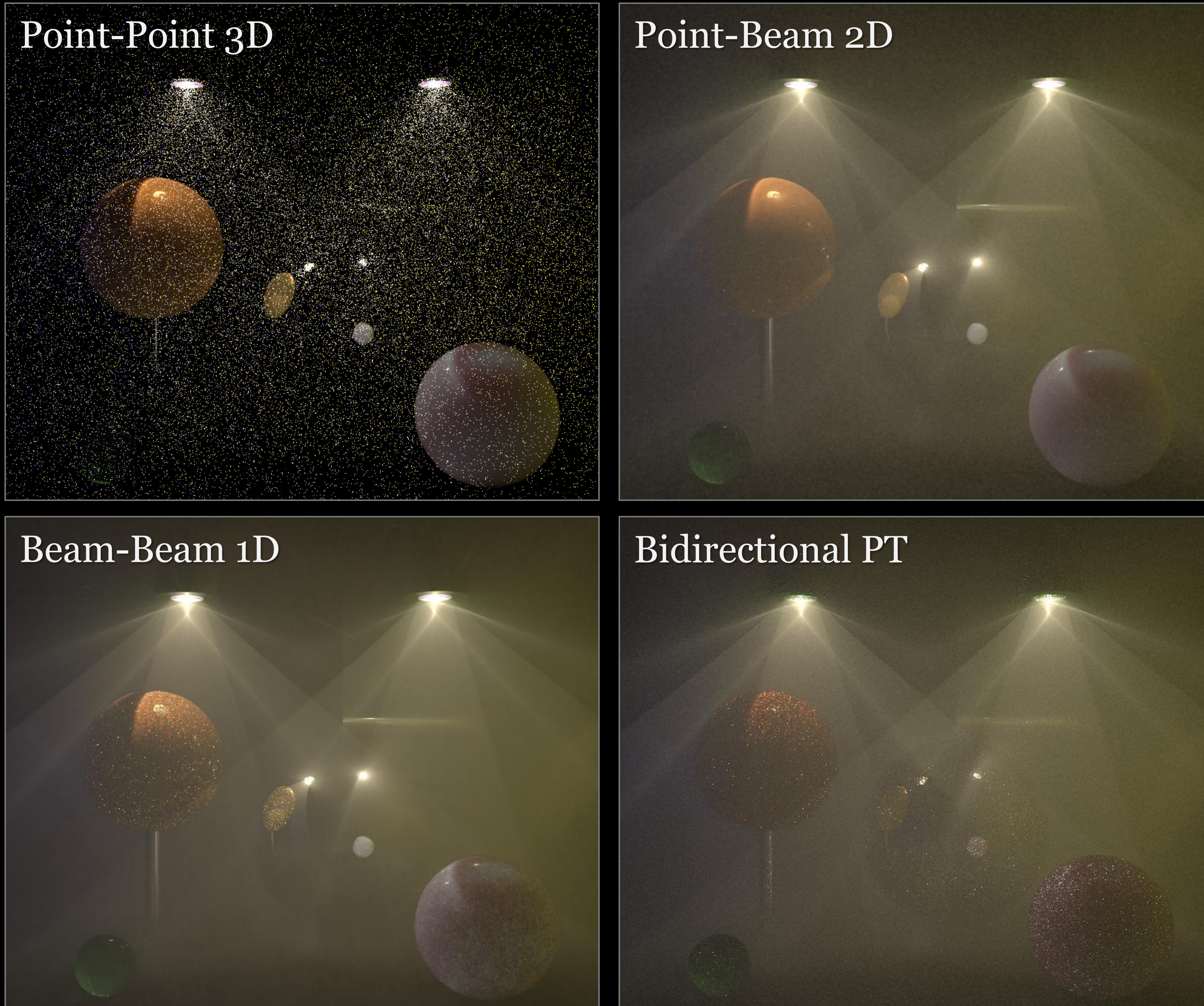

Full transport
rare, fwd-scattering fog
back-scattering
high albedo
back-scattering

Medium transport only

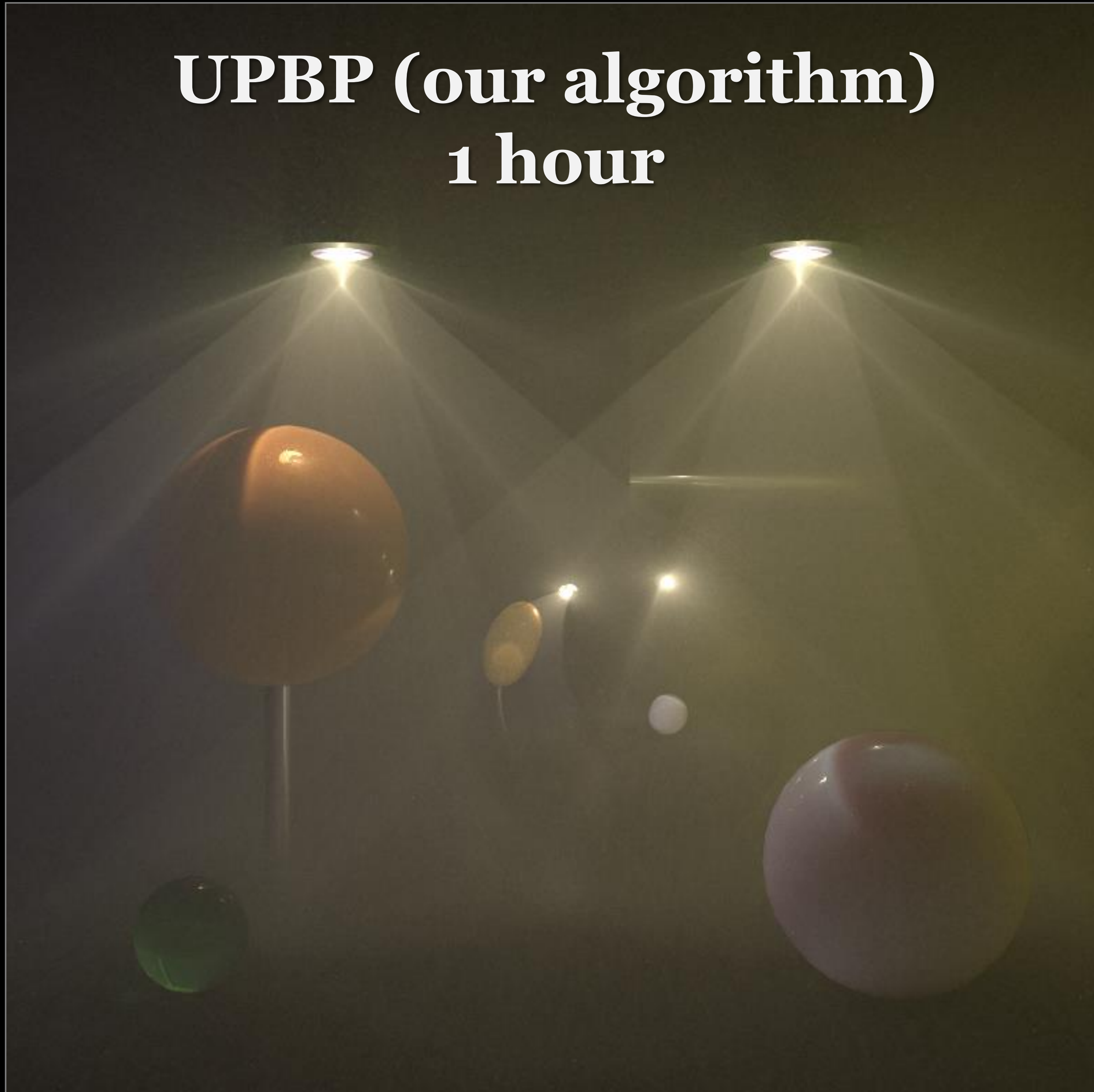
Previous work comparison, 1 hr



Previous work comparison, 1 hr



Weighted contributions


UPBP (our algorithm)
1 hour